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Laminar convection cells at high Rayleigh number

By P. WESSELING†
Jet Propulsion Laboratory, Pasadena, California

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The asymptotic behaviour for large Rayleigh number and Prandtl number of $O(1)$ of two-dimensional convection cells in a fluid between horizontal plates heated from below has been discussed by Pillow (1952) and more recently by Robinson (1967). The flow models derived by Pillow and Robinson differ from each other. The purpose of the present paper is to point out the likelihood of an inconsistency in Robinson's flow model, and to discuss qualitatively a few hitherto unnoticed features of the flow.

1. Introduction

The asymptotic behaviour for large Rayleigh number and Prandtl number of $O(1)$ of two-dimensional convection cells has recently been discussed by Robinson (1967).

Robinson chooses to make the equations dimensionless in the following way:

$$\mathbf{r}^* = d\mathbf{r}, \quad \mathbf{v}^* = \frac{\kappa}{d}\mathbf{v}, \quad t^* = \frac{d^2}{\kappa}t, \quad T^* = T_0 + \Delta T\theta, \quad p^* - \rho_0 g z^* = \frac{\kappa\nu\rho_0}{d^2}p, \tag{1}$$

where asterisks denote dimensional variables, $\mathbf{r}^*$ and $\mathbf{v}^*$ are the position and the velocity vector, d is the distance between the horizontal plates, κ is the thermal conductivity, ν is the kinematic viscosity, T^* is the temperature, T_0 is the average of the temperatures of the plates, $2\Delta T$ is the temperature difference between the plates, p^* is the pressure, ρ_0 is the average density, g is the acceleration of gravity, and z^* is the vertical co-ordinate.

Although the Rayleigh number (to be defined shortly) tends to infinity, the flow is assumed to be steady. The dimensionless equations are, using the Boussinesq approximation:

$$\operatorname{div} \mathbf{v} = 0, \tag{2}$$

$$(\mathbf{v} . \nabla)\theta = \nabla^2\theta, \tag{3}$$

$$\frac{1}{\sigma}(\mathbf{v} . \nabla)\mathbf{v} = -\nabla p + R\theta\mathbf{k} + \nabla^2\mathbf{v}. \tag{4}$$

Taking the curl of (4) gives

$$\frac{1}{\sigma}(\mathbf{v} . \nabla)\eta = -R\frac{\partial\theta}{\partial x} + \nabla^2\eta. \tag{5}$$

In these equations, $\mathbf{k}$ is the unit vector in the positive z-direction (upwards), $R = g\alpha\Delta T d^3/\kappa\nu$ (where α is the thermal expansion coefficient) is the Rayleigh

† Present address: National Aerospace Laboratory, Amsterdam, Netherlands.

number, $\sigma = \nu/\kappa$ is the Prandtl number, and η is the vorticity. It is assumed that the motion is two-dimensional. Let the (x, z)-plane be parallel to the velocity vector, so that the vorticity is given by

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}.$$

Figure 1 represents a convection cell. Three different cases are studied by Robinson (1967): (i) the full boundary $ABCD$ rigid; (ii) fluid both inside and outside $ABCD$; (iii) AD and BC rigid, fluid at both sides of AB and CD. Only the third case will be considered here.

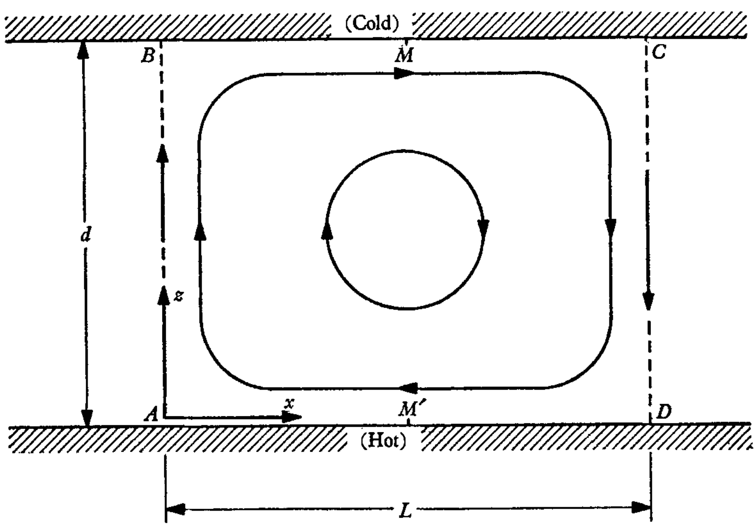

FIGURE 1. A two-dimensional convection cell.

Equation (2) guarantees the existence of a stream function ψ such that

$$u = \partial\psi/\partial z, \quad w = -\partial\psi/\partial x. \tag{6}$$

The boundary conditions along AB and CD are

$$\psi = 0, \quad \partial w/\partial x = 0, \quad \partial\theta/\partial x = 0. \tag{7}$$

The boundary conditions along AD and BC are

$$\psi = 0, \quad u = 0, \quad \theta = \begin{Bmatrix} +1 & \text{along } AD, \\ -1 & \text{along } BC. \end{Bmatrix} \tag{8}$$

2. Description of Robinson's flow model

The flow in the interior of the cell is taken to be non-dissipative, so that temperature and vorticity are constant along streamlines. As shown by Pillow (1952), $\theta = 0$ in the interior. According to a theorem due to Prandtl (1904), the vorticity equals η_0, a constant, in the interior.

For the regions close to AB and BC, Robinson substitutes the following asymptotic expansions in the equations:

$$AB: \quad \left\{ \begin{aligned} &\psi = R^b\phi_0(x,z) + R^{b-ma}\phi^{(1)}_{BL}(\xi,z) + R^{b-ma}\phi^{(a)}_1(x,z) + \dots, \\ &\theta = \theta^{(1)}_{BL}(\xi,z) + \dots, \\ &\xi = xR^a. \end{aligned} \right\} \tag{9}$$

$$BC: \quad \left\{ \begin{aligned} &\psi = R^b\phi_0(x,z) + R^{b-m'a'}\phi^{(3)}_{BL}(x,\zeta) + R^{b-m'a'}\phi^{(b)}_1(x,z) + \dots, \\ &\theta = \theta^{(3)}_{BL}(x,\zeta) + \dots, \\ &\zeta = (z-d)\,R^{a'}. \end{aligned} \right\} \tag{10}$$

Robinson determines the exponents a, b, m, a' and m' by balancing the heat convection and heat diffusion terms in (3), the vorticity convection, vorticity diffusion and vorticity production terms in (5), heat conducted through the walls and heat convected through the vertical layers, and by using some of the boundary conditions along AB and BC. The boundary conditions used are: $\partial w/\partial x = 0$ along AB, and $u = 0$ along BC. This gives

$$R^b \left.\frac{\partial^2\phi_0}{\partial x^2}\right|_{x=0} + R^{b-ma+2a} \left.\frac{\partial^2\phi^{(1)}_{BL}}{\partial \xi^2}\right|_{\xi=0} = 0, \tag{11}$$

$$R^b \left.\frac{\partial\phi_0}{\partial z}\right|_{z=d} + R^{b-m'a'+a'} \left.\frac{\partial\phi^{(3)}_{BL}}{\partial \zeta}\right|_{\zeta=0} = 0. \tag{12}$$

The following result is obtained:

$$a = \tfrac{1}{3}, \quad b = \tfrac{2}{3}, \quad m = 2, \quad a' = \tfrac{1}{3}, \quad m' = 1. \tag{13}$$

The values $m = 1$, $m' = 2$ listed on p. 581 of Robinson's paper seem to be misprints. Equation (9) of Robinson's article gives explicitly $m = 2$: this follows from (11) above. The value $m' = 1$ follows from (12) above, which represents the no-slip condition along the horizontal wall. Also, $m' = 1$ corresponds to the statement made on p. 581 of Robinson's article, that the viscous torque is $O(R)$ (see equation (16)).

It is instructive to let R tend to infinity by letting the viscosity tend to zero with all other parameters fixed. Robinson's flow model then gives a viscous region with thickness of order $\nu^{\frac{2}{3}}$, and (dimensional) velocities in the inviscid interior of order $\nu^{-\frac{1}{3}}$.

3. Discussion of Robinson's flow model

The total (dimensional) buoyancy force working on a vertical boundary layer is given by

$$\text{buoyancy force} = \rho_0 g\alpha d^2 \Delta T R^{-a} \int_0^1 dz \int_0^\infty \theta\, d\xi. \tag{14}$$

The total (dimensional) frictional force working on a horizontal boundary layer is given by

$$\text{frictional force} = \rho_0 g\alpha d^2 \Delta T R^{-1} \int_0^{L/d} \left.\frac{\partial u}{\partial z}\right|_{\zeta=0} dx, \tag{15}$$

where L is the length of a cell.

With the limit process defined by equation (13) one finds

$$\left.\begin{aligned} R^{-a}\int_0^1 dz \int_0^\infty \theta \, d\xi &= O(R^{-\frac{1}{3}}), \\ \int_0^{L/d} \frac{\partial u}{\partial z}\bigg|_{\zeta=0} dx &= O(R). \end{aligned}\right\} \tag{16}$$

Hence the friction is an order of magnitude larger than the buoyancy force. According to Robinson, the viscous torque working on a convection cell is balanced by the 'second-order pressure torque' (Robinson 1967, p. 581).

With the limit process defined by equation (13), the equation of motion for the horizontal boundary layers is found to be, to leading order,

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = U\frac{dU}{dx} + \sigma\frac{\partial^2 u}{\partial z^2}, \tag{17}$$

where U is the velocity at the cell boundary as given by the Euler solution (the Euler velocity). The dependent variables and their derivatives which occur in (17) are of various orders of magnitude. It is convenient to transform (17) into an equation in which all dependent variables and their derivatives remain $O(1)$ as $R \to \infty$. From (10) and (13) it follows that this can be done by means of the following substitutions:

$$u = \bar{u}R^{\frac{2}{3}}, \quad w = \bar{w}R^{\frac{1}{3}}, \quad U = \bar{U}R^{\frac{2}{3}}, \quad x = \bar{x}, \quad z = \bar{z}R^{-\frac{1}{3}}. \tag{18}$$

With these substitutions, (17) transforms into itself:

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}} = \bar{U}\frac{d\bar{U}}{d\bar{x}} + \sigma\frac{\partial^2 \bar{u}}{\partial \bar{z}^2}. \tag{19}$$

The boundary conditions are, for the lower horizontal boundary layer,

$$\bar{u}(\bar{x}, 0) = 0, \quad \bar{u}(\bar{x}, \infty) = \bar{U}(\bar{x}). \tag{20}$$

The pressure gradient $-\bar{U}d\bar{U}/d\bar{x}$ is known from the Euler solution as computed by Pillow (1952) and by Robinson (1967). The pressure gradient is negative downstream from B (see figure 1) up to the point M, which is equidistant from B and C. Downstream of M, the pressure gradient is positive; at C the pressure is the same as at B. The pressure gradient $-\bar{U}d\bar{U}/d\bar{x}$ is $O(1)$. The behaviour of the solutions of (19) with boundary conditions (20) under these circumstances is well known, both from experience and from theoretical considerations. It can be shown theoretically (see for, instance, Kaplun's argument, as formulated by Lagerstrom (1964)) that, at a short distance from the pressure-minimum at M, at an appreciable distance from the corner C, the skin-friction vanishes. Experimentally, it is observed that at or near the point of vanishing skin-friction the flow separates from the wall. Separation need not occur at a short distance downstream from M in the exceptional case that the initial conditions are such that the boundary layer starts out (at B and at D) with a velocity overshoot (i.e. negative momentum thickness), like a wall-jet. In this case the boundary layer carries more momentum and can negotiate a larger pressure rise than without a velocity overshoot; it is necessary that the velocity overshoot be of the same order of magnitude as the Euler velocity.

The possibility of boundary-layer separation is not taken into account by Robinson, or by Pillow. This in itself does not invalidate their flow models. The orders of magnitude of the velocity and the thickness of the boundary layers as deduced by Robinson or by Pillow might still be correct. Furthermore, as will be shown later in this paper, with Pillow's flow model the horizontal boundary layers have velocity overshoots (this was not noticed by Pillow), so that separation may be postponed till a short distance from the corners and perhaps does not play an important role.

In Robinson's flow model the horizontal boundary layers cannot have velocity overshoots which are of the same order of magnitude as the Euler velocity. This follows from the properties of the vertical shear layers. According to (9) and (13), the vertical velocity in the shear layer AB is given by

$$w = W - R^{\frac{1}{3}}\partial\phi^{(1)}_{BL}/\partial\xi - R^{\frac{1}{3}}\partial\phi_1^{(a)}/\partial x,$$

where the Euler velocity W is given by

$$W = -R^{\frac{2}{3}}\partial\phi_0/\partial x.$$

The difference between the velocity in the shear layer and the Euler velocity is of smaller order of magnitude ($O(R^{\frac{1}{3}})$) than the Euler velocity ($O(R^{\frac{2}{3}})$), so that the vertical shear layers do not impart enough momentum to the horizontal boundary layers at B and D to generate a velocity overshoot which is of the same order of magnitude as the Euler velocity. Hence, in Robinson's flow model separation is expected to occur at a short distance downstream from M and M'.

It does not seem possible to modify Robinson's flow model in order to take separation into account, without arriving at an inconsistency. This is suggested by the following mathematical argument, which is essentially based upon the fact that the viscous forces are of a larger order of magnitude than the buoyancy forces, and on the conjecture that near the point of vanishing skin friction, at a distance of $O(1)$ from the corners, a streamline leaves the wall; as stated before, this is what happens in experiments. When applying the following argument to Pillow's flow model with non-negligible separation (i.e. separation at a distance of $O(1)$ from the corners), no inconsistency is found.

The exact dimensionless equations of motion may be written in the following form:

$$\nabla(H - (1/\sigma)\mathbf{v}\times\mathbf{j}\eta) = -\operatorname{curl}\mathbf{j}\eta + R\theta\mathbf{k}, \tag{21}$$

where $H = (1/2\sigma)(u^2+w^2)+p$ is the total head, $\eta = (\partial u/\partial z)-(\partial w/\partial x)$ is the vorticity, and $\mathbf{j}$ is the unit vector in the Y-direction.

Multiplication of (21) with a vector element of a streamline $d\mathbf{s}$ gives

$$(\partial H/\partial s)\,ds = -d\mathbf{s}\,.\operatorname{curl}\mathbf{j}\eta + R\theta\, d\mathbf{s}\,.\mathbf{k}.$$

Single-valuedness of H requires that along every streamline the following equation holds:

$$\oint d\mathbf{s}\,.\operatorname{curl}\mathbf{j} - R\oint \theta\, d\mathbf{s}\,.\mathbf{k} = 0. \tag{22}$$

(The streamlines are assumed to be closed.) An approximation to the exact solution must satisfy (22) within the order of approximation. It will be shown that (22) cannot be satisfied by Robinson's flow model, modified to take separation into account.

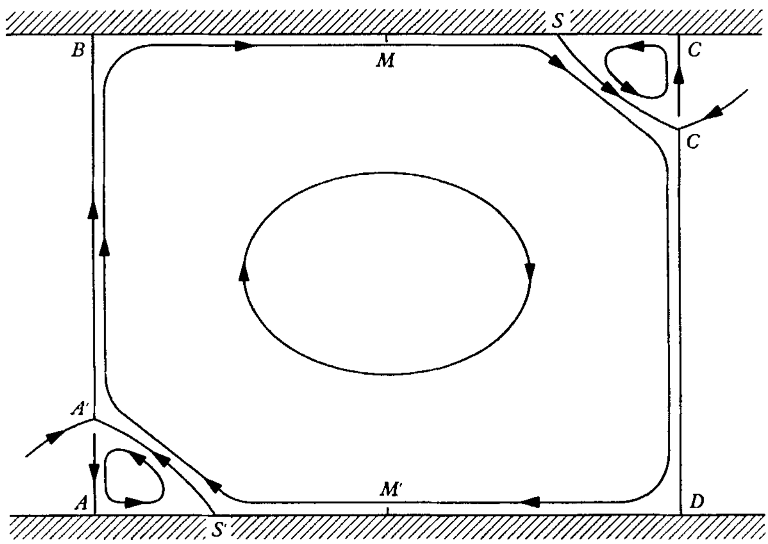

FIGURE 2. A two-dimensional convection cell with flow separation.

Consider a streamline in the viscous region close to the cell-boundary $A'BSC'DS'A'$ (see figure 2). The following equality holds:

$$d\mathbf{s}\,.\,\mathrm{curl}\,\mathbf{j}\eta = -(\partial\eta/\partial n)\,ds,$$

where the co-ordinates (s, n) measure distance along and perpendicular to (positive outward) the streamline under consideration. With Robinson's flow model, as defined by (13), the order of magnitude of $\partial\eta/\partial n$ is found to be as follows:

$$\text{Close to } A'B\text{:} \quad \partial\eta/\partial n = O(R). \tag{23}$$

$$\text{Close to } BS\text{:} \quad \partial\eta/\partial n = O(R^{\frac{4}{3}}). \tag{24}$$

$$\text{Close to } SC'\text{:} \quad \partial\eta/\partial n \leqslant O(R^{\frac{4}{3}}). \tag{25}$$

$$\text{Close to } C'D\text{:} \quad \partial\eta/\partial n = O(R). \tag{26}$$

$$\text{Close to } DS'\text{:} \quad \partial\eta/\partial n = O(R^{\frac{4}{3}}). \tag{27}$$

$$\text{Close to } S'A'\text{:} \quad \partial\eta/\partial n \leqslant O(R^{\frac{4}{3}}). \tag{28}$$

Robinson's flow model does not give the order of magnitude of $\partial\eta/\partial n$ in the separated layers, because separation is not taken into account. If one assumes that the order of magnitude of the velocity in the separated layer is not larger than the largest velocity which occurs in Robinson's model (namely $O(R^{\frac{2}{3}})$), then the thickness of the separated layers cannot be of smaller order of magnitude than the thickness of the horizontal boundary layers (which is $O(R^{-\frac{1}{3}})$), because this would cause the vorticity flux to be discontinuous in the regions where the flow separates, which is impossible, because there are no vorticity sources or sinks of infinite strength (like for instance the leading edge of a flat plate) in these regions. The inequalities (25) and (28) follow.

If the equality signs hold in (25) and (28) equation (22) is, to leading order,

$$\int_B^S \frac{\partial\eta}{\partial n}ds + \int_S^{C'} \frac{\partial\eta}{\partial n}ds + \int_D^{S'} \frac{\partial\eta}{\partial n}ds + \int_{S'}^{A'} \frac{\partial\eta}{\partial n}ds = 0. \tag{29}$$

For a streamline very close to the cell boundary, the following approximate equalities hold (using (17)).

$$\text{Along } BS\text{: } \partial\eta/\partial n \cong \partial^2 u/\partial z^2 \cong -(1/\sigma)\, UdU/dx. \tag{30}$$

$$\text{Along } DS'\text{: } \partial\eta/\partial n \cong -\partial^2 u/\partial z^2 \cong (1/\sigma)\, UdU/dx. \tag{31}$$

Hence

$$\int_B^S \frac{\partial\eta}{\partial n}ds + \int_D^{S'} \frac{\partial\eta}{\partial n}ds = -\frac{1}{\sigma}\int_B^S U\frac{dU}{ds}ds + \frac{1}{\sigma}\int_D^{S'} U\frac{dU}{dx}ds$$

$$= -\frac{1}{2\sigma}[U^2(S) + U^2(S')] = -\frac{1}{\sigma}U^2(S), \tag{32}$$

where $U(S) \neq 0$, because separation must take place upstream of a stagnation point of the Euler solution.

The vorticity, defined as $\eta = (\partial u/\partial z) - (\partial w/\partial x)$, is positive inside the cell $A'BSC'DS'A'$, and negative inside the separation bubbles SCC' and $S'AA'$. Therefore, $\partial\eta/\partial n < 0$ along SC' and $S'A'$.

Hence (29) cannot be satisfied.

If the inequality signs hold in (25) and (28), equation (22) is to leading order

$$\int_B^S \frac{\partial\eta}{\partial n}ds + \int_D^{S'} \frac{\partial\eta}{\partial n}ds = 0. \tag{33}$$

From (32) it follows that (33) is not satisfied.

Hence Robinson's leading-order solution, modified to take separation into account, does not satisfy equation (22), and is therefore not single-valued, so that Robinson's modified leading-order solution cannot represent the asymptotic behaviour of the flow. As will be shown in the next section, Pillow's flow model can satisfy the single-valuedness condition (equation (22)). The underlying reason is that in Pillow's flow model the viscous forces are of the same order of magnitude as the buoyancy forces.

The point at which Robinson's study takes a course which does not lead to Pillow's flow model is the conclusion drawn from equation (11), namely $m = 2$. The correct conclusion is $m \leqslant 2$, because the boundary-layer solution can always be made to satisfy the following boundary condition:

$$\left.\frac{\partial^2 \phi_{BL}^{(1)}}{\partial \xi^2}\right|_{\xi=0} = 0. \tag{34}$$

Of course Robinson's choice of $m = 2$ is permitted, but it is not required, and (this is our main point) leads to the serious difficulty of insisting that the horizontal boundary layers suffer only a negligible separation, in spite of being subject to an $O(1)$ adverse pressure gradient over half their length.

The remainder of this paper will be devoted to a discussion of Pillow's flow model, and to a comparison of Robinson's and Pillow's results with numerical calculations by Fromm (1965).

4. Discussion of Pillow's flow model

It is convenient to introduce new dimensionless variables, which turn out to remain $O(1)$ (although not some of their derivatives) as $R \to \infty$.

The maximum possible buoyancy force which can work on a fluid element is $\rho_0 g\alpha\Delta T$. Taking $g\alpha\Delta T$ as the unit of acceleration, and d as the unit of length, one obtains $[g\alpha d\Delta T]^{\frac{1}{2}}$ as the velocity unit. With

$$\mathbf{r}^* = d\mathbf{r}, \quad \mathbf{v}^* = [g\alpha d\Delta T]^{\frac{1}{2}}\tilde{\mathbf{v}}, \quad T^* = T_0 + \Delta T\theta, \quad p^* - \rho_0 g z^* = \rho_0 g\alpha d\Delta T\tilde{p},$$

the Boussinesq equations are

$$(\tilde{\mathbf{v}}\,.\,\nabla)\,\tilde{\mathbf{v}} = -\nabla\tilde{p} + \theta\mathbf{k} + \sigma^{\frac{1}{2}}R^{-\frac{1}{2}}\nabla^2\tilde{\mathbf{v}}, \tag{35}$$

$$(\tilde{\mathbf{v}}\,.\,\nabla)\theta = \sigma^{-\frac{1}{2}}R^{-\frac{1}{2}}\nabla^2\theta. \tag{36}$$

$$\operatorname{div}\tilde{\mathbf{v}} = 0. \tag{37}$$

It seems reasonable to assume that as $R \to \infty$ the diffusive terms will be negligible, except close to the cell boundaries, so that in the interior of the cell the Euler equations hold. Assuming that the streamlines are closed, Pillow (1952) has shown that $\theta = 0$ in the interior. According to Prandtl's (1904) theorem, the vorticity equals $\tilde{\eta}_0$, a constant, in the interior, so that the Euler solution satisfies the following equation:

$$\nabla^2\tilde{\phi}_0 = \tilde{\eta}_0, \tag{38}$$

where $\tilde{\phi}_0$ is the leading term in the asymptotic expansion for the stream function of the Euler solution. The constant $\tilde{\eta}_0$ follows from the requirement that a single-valued solution for the thin viscous region along the cell boundary be possible. In general, it is very difficult to actually determine $\tilde{\eta}_0$; this has been done only for the simple case where the cell has a circular cross-section (Batchelor 1956; Feynman & Lagerstrom 1956).

A solution of (38) has been constructed in the form of a series by Pillow (1952) and Robinson (1967). It is noteworthy that the Euler solution has stagnation points in the corners.

The equations which describe the flow in the thin boundary layers along the cell boundaries (henceforth to be called the Prandtl equations)† must have the following properties: (i) diffusive terms must be present, so that the boundary conditions at the cell boundaries can be satisfied; (ii) in order to make matching possible, the Euler and Prandtl expansions must have a 'domain of overlap' (Kaplun & Lagerstrom 1957; Kaplun 1957). It is highly plausible that a sufficient condition for this requirement to be satisfied is that the Euler and Prandtl equations have a 'domain of overlap'. This is the case if the convective term and the pressure term are present in the Prandtl equation,

† The term 'Prandtl equations' is used in a generalized sense. It is not *a priori* clear that the Prandtl equations will be identical with Prandtl's classical boundary-layer equation.

possibly in simplified form; (iii) the viscous term must be of the same order of magnitude as the buoyancy term, in order to make it possible to satisfy the single-valuedness condition, equation (22); (iv) given the fact that the convective term is present, the Prandtl equations must also contain a pressure term. This can be seen as follows:

Taking the divergence of (35) gives, in the absence of the pressure term,

$$2\left(\frac{\partial \tilde{u}}{\partial x}\right)^2 + 2\left(\frac{\partial \tilde{u}}{\partial z}\right)^2 = 2\tilde{\eta}\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \theta}{\partial z}. \tag{39}$$

In the outer parts of the viscous region, $\partial\theta/\partial z \to 0$, $\tilde{\eta} \to \tilde{\eta}_0$. Wherever $\partial\tilde{u}/\partial z$ changes sign (which is required by the matching with the Euler solution) the right-hand side of (39) changes sign; however, the left-hand side is positive semi-definite. Hence a Prandtl solution which matches with the Euler solution does not exist if the Prandtl equations do not contain a pressure term.

If it turns out that no limit process can be found which gives Prandtl equations which satisfy the four requirements listed above, additional subregions, each with its own limit process, must be introduced, and the assumption of inviscid flow in the interior must be reconsidered. On the other hand, it is not claimed that the set of four requirements just given is sufficient to select a unique limit process.

The reader may easily convince himself, that the following limit process satisfies the four requirements: (*a*) the co-ordinate normal to the cell boundary is stretched proportionally to $R^{\frac{1}{4}}$, i.e. $x = R^{-\frac{1}{4}}\xi$ along AB (or $A'B$ if separation takes place), $z-d = R^{-\frac{1}{4}}\zeta$ along BC (or BS), etc. This corresponds to a boundary-layer thickness of $O(\nu^{\frac{1}{2}})$; (*b*) all other quantities remain $O(1)$.

The Prandtl equations are found to be, in unstretched variables, in the vertical layers:

$$\tilde{u}\frac{\partial \tilde{w}}{\partial x} + \tilde{w}\frac{\partial \tilde{w}}{\partial z} = \tilde{W}\frac{d\tilde{W}}{dz} + \theta + \sigma^{\frac{1}{2}}R^{-\frac{1}{2}}\frac{\partial^2 \tilde{w}}{\partial x^2}, \tag{40}$$

$$\operatorname{div}\tilde{\mathbf{v}} = 0, \tag{41}$$

$$\tilde{u}\frac{\partial \theta}{\partial x} + \tilde{w}\frac{\partial \theta}{\partial z} = \sigma^{-\frac{1}{2}}R^{-\frac{1}{2}}\frac{\partial^2 \theta}{\partial x^2}; \tag{42}$$

in (the unseparated portions of) the horizontal layers:

$$\tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{w}\frac{\partial \tilde{u}}{\partial z} = \tilde{U}\frac{d\tilde{U}}{dx} + \sigma^{\frac{1}{2}}R^{-\frac{1}{2}}\frac{\partial^2 \tilde{u}}{\partial z^2}. \tag{43}$$

$$\operatorname{div}\tilde{\mathbf{v}} = 0, \tag{44}$$

$$\tilde{u}\frac{\partial \theta}{\partial x} + \tilde{w}\frac{\partial \theta}{\partial z} = \sigma^{-\frac{1}{2}}R^{-\frac{1}{2}}\frac{\partial^2 \theta}{\partial z^2}. \tag{45}$$

Here $\tilde{U}$ and $\tilde{W}$ are the Euler velocities at the horizontal and vertical cell boundaries.

Equations (40)–(45) were obtained by Pillow (1952), who constructed an approximate solution by replacing the convective velocity by a known function.

The single-valuedness condition (22) can be rewritten as

$$\sigma^{\frac{1}{2}}R^{-\frac{1}{2}}\oint\frac{\partial\tilde{\eta}}{\partial n}ds + \oint \theta\, d\mathbf{s}\,.\,\mathbf{k} = 0, \tag{46}$$

where $\tilde{\eta}$ is the vorticity with the scaling adopted at the beginning of this section. With Pillow's flow model, $\partial\tilde{\eta}/\partial n = O(R^{\frac{1}{2}})$ in the viscous region (see the limit process definition preceding (40)), so that both terms in (46) are $O(1)$. Hence, Pillow's flow model can indeed satisfy the single-valuedness condition.

An interesting feature of the solution of (40)–(45), which was not mentioned by Pillow, can be brought out by the following qualitative discussion.

In the upward-moving vertical layer $\theta > 0$, because the flow has just passed close to the hot lower wall. Equation (40) shows that the buoyancy force generates a velocity overshoot in the upward-moving layer. For instance, the difference between the kinetic energy in the centre ($\xi = 0$) of the upward-moving layer and the kinetic energy just outside the layer satisfies the following equation:

$$\frac{d}{dz}[\tfrac{1}{2}\tilde{w}^2(0,z) - \tfrac{1}{2}\tilde{W}^2(z)] = \theta(0,z) + \sigma^{\frac{1}{2}}R^{-\frac{1}{2}}\frac{\partial^2\tilde{w}}{\partial x^2}\bigg|_{(0,z)}. \tag{47}$$

The retarding viscous force cannot completely balance the buoyancy force, because the viscous force becomes small when the difference between $w(0,z)$ and $W(z)$ becomes small.

Therefore, the vertical layer develops into a jet as it moves upward, impinging on the upper wall and generating a velocity overshoot in the upper horizontal layer which resembles a wall-jet. The velocity difference between the vertical jet and the Euler flow is $O(1)$. Hence, the velocity overshoot in the horizontal layer will also be $O(1)$. Similarly, the downward-moving vertical boundary layer develops into a jet, owing to the negative buoyancy force which is acting in the downward-moving layers, so that there is also a velocity overshoot of $O(1)$ in the lower horizontal boundary layer. Hence, unlike the horizontal boundary layers in Robinson's flow model, the horizontal boundary layers do not necessarily separate at a short distance downstream of the pressure minima at M and M' (figure 2). But, depending on the strength of the jets, separation might still take place at a distance of $O(1)$ from the corner.

Equation (47) leads one to expect that the vertical jets will become more powerful as the Rayleigh number increases. When R becomes larger, $\frac{1}{2}\tilde{w}^2 - \frac{1}{2}\tilde{W}^2$ will increase, so that $\partial^2\tilde{w}/\partial x^2|_{0,z}$ increases, until the viscous term is again balanced against the buoyancy term and the inertia term.

Robinson's and Pillow's flow model, and the qualitative predictions made in the present paper, will now be compared with the numerical results obtained by Fromm (1965).

Pillow's (1952) approximate calculations result in the following relation between the Nusselt number and the Rayleigh number:

$$N = 0{\cdot}86R^{\frac{1}{4}}, \tag{48}$$

where $$N = \int_0^{L/d} \frac{\partial\theta}{\partial z}\bigg|_{\zeta=0} dx.$$

The cells are assumed to be square and $\sigma = 1$. The fact that N is proportional to $R^{\frac{1}{4}}$ in Pillow's flow model does not depend on Pillow's approximate calculations, but follows directly from the limiting process which was used to obtain equations

(40)–(45). The approximate calculations merely serve to determine the constant of proportionality.

According to Robinson (1967, p. 594) the relation between N and R obtained from the numerical solution of the exact equations constructed by Fromm (1965) can be approximately represented by

$$N \cong 0{\cdot}19R^{0{\cdot}28}. \tag{49}$$

Fromm (1965, p. 1763) states that

$$N \sim R^{0{\cdot}296}. \tag{50}$$

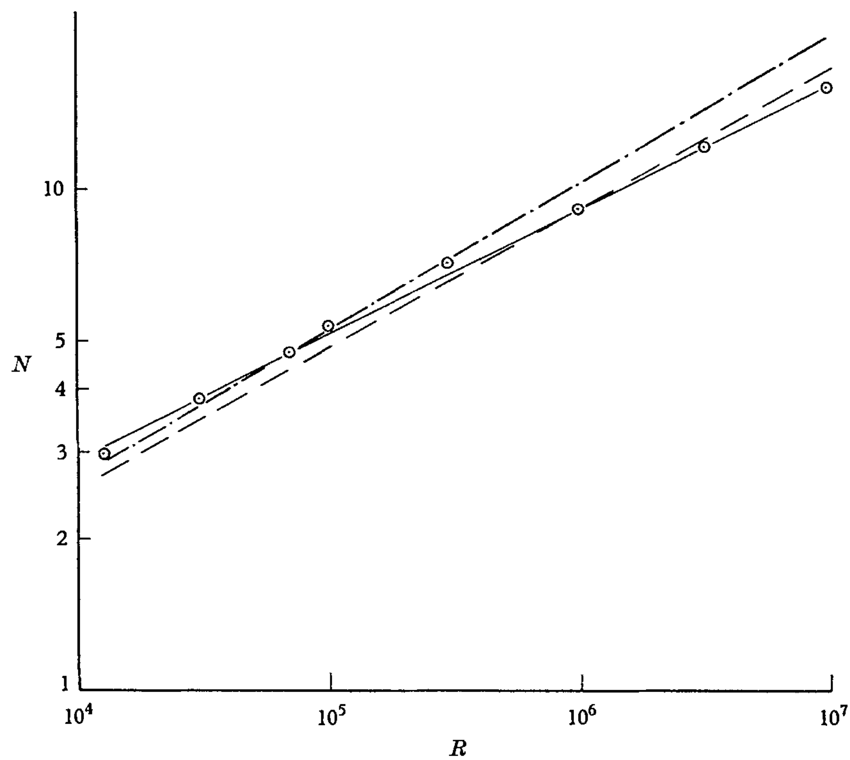

FIGURE 3. Dependence of Nusselt number on Rayleigh number. ⊙, Fromm's numerical results; ———, $N = 0{\cdot}285R^{\frac{1}{4}}$; – – –, $N = 0{\cdot}19R^{0{\cdot}28}$; –·–·–, $N \sim R^{0{\cdot}296}$.

According to Fromm (p. 1765), N drops off for $R \gtrsim 3\times 10^5$ owing to boundary-layer separation.

As may be seen in figure 3, a good fit to Fromm's data is also obtained with

$$N = 0{\cdot}285R^{\frac{1}{4}}. \tag{51}$$

The inaccuracy in bringing over the points representing Fromm's results in figure 3 from Fromm's (1965) rather small-sized figure 7 is estimated to be of the order of 1/50 of the horizontal unit in figure 3.

Robinson's (1967, p.594) best estimate is

$$N = 0{\cdot}15R^{\frac{1}{3}}. \tag{52}$$

Again, the fact that N is proportional to $R^{\frac{1}{3}}$ follows directly from Robinson's limiting process, defined by (13).

Figure 3 shows that Fromm's results are in favour of limit processes which make N proportional to $R^{\frac{1}{4}}$.

The discrepancy between Robinson's result (52) and Fromm's results (represented in figure 3) is explained by Robinson (1967, pp. 594, 595) in the following way. Robinson suggests that the Pillow model gives a smaller heat transfer than the Robinson model for $R \gtrsim 4 \times 10^5$, but a larger heat transfer for $R \lesssim 4 \times 10^5$. Furthermore, Robinson assumes that perhaps the flow behaves in such a way as to maximize the heat transfer, so that the Pillow model is realized for $R \lesssim 4 \times 10^5$, and the Robinson model for $R \gtrsim 4 \times 10^5$. It is further argued by Robinson (in agreement with a private communication by Fromm) that for $R \gtrsim 3 \times 10^5$ Fromm's calculations give a Nusselt number which is too low, owing to insufficient resolution of the boundary layer. In other words, in the region where Robinson's model is valid, Fromm's results are inaccurate. Note that, if the drop in Nusselt number for $R \gtrsim 3 \times 10^5$ cited by Robinson and Fromm is due to flow separation, as suggested by Fromm (1965, p. 1765), this behaviour of N cannot be termed an inaccuracy.

Be this as it may, Fromm's results seem to support the proportionality of N with $R^{\frac{1}{4}}$ as predicted by Pillow.

The large error in Pillow's prediction of the proportionality constant may be ascribed to the mistake in Pillow's analysis which was found by Robinson (1967, p. 583), to the fact that Pillow's calculations do not simulate the velocity overshoots in the viscous region, and to the fact that Pillow's calculations are (necessarily) approximate.

Fromm's isodine patterns (Fromm 1965, figure 9) do not show the type of behaviour of the vorticity which one would expect if the vertical jets predicted in the present paper were present. There do not seem to be large vorticity gradients across the vertical layers. However, with the finer resolution of the pattern given in Fromm's figure 8, some vorticity variation across the vertical layers begins to appear. It seems possible that for the range of Rayleigh numbers studied by Fromm the vertical jets are not yet well developed. The horizontal boundary layers separate at a large distance from the corners, upstream from the mid-point even, but as R increases the separation point moves downstream (Fromm 1965, figure 9). This may be caused by the vertical jets which grow stronger as R increases. Fromm's streamline patterns seem to show some evidence of the vertical jets predicted in the present paper. The streamlines in the vertical layers are crowded together, and more so as the opposite wall is approached (Fromm 1965, figure 9).

It is of interest that Fromm's results show that the flow separates at the horizontal walls (see the streamline patterns in Fromm's figure 9), since our deduction of an inconsistency in the Robinson flow model hinges on the occurrence of boundary-layer separation.

The fact that flow separation occurs upstream of the mid-point between the corners seems to indicate that the separation influences the Euler flow in the centre of the cell, so that the pressure minimum moves upstream.

It should be kept in mind that for $R \gtrsim 10^6$ Fromm's results oscillate in time (in a predictable way), and that Fromm's figure 9 represents just one of the several flow patterns that may occur during one oscillation. Furthermore, Fromm (1965, p. 1763) suggests that for $R \gtrsim 10^7$ the boundary layers are insufficiently resolved in his calculations; Robinson (1967, p. 595) suggests that this may already be the case for $R \gtrsim 3 \times 10^5$.

This comparison of Robinson's flow model, Pillow's flow model and some qualitative features discussed in the present paper with Fromm's calculations may be summarized as follows: (i) Fromm's results do not support the Robinson model. Robinson suggests that Fromm's results are inaccurate in the region of validity of the Robinson model; (ii) Fromm's data support the relationship $N \sim R^{\frac{1}{4}}$, predicted by Pillow; (iii) Fromm's results, by showing flow separation, support the argument given in the present paper to point out that it is likely that the Robinson flow model is inconsistent; (iv) Fromm's results do not constitute a strong argument against the occurrence of the vertical jets predicted in the present paper, because they are ambiguous on this point.

Taylor vortex cells between rotating concentric cylinders are qualitatively similar to two-dimensional convection cells. Approximate calculations by the present author (1967) of strongly developed Taylor vortex flow, in which the velocity overshoots in the viscous regions were simulated, resulted in torque predictions which are 15–30 % higher than experimentally observed values. It was also predicted that the dimensionless torque should be inversely proportional to the one-quarter power of the Taylor number. This was found to be approximately in agreement with experiment in one case, and exactly in agreement in another case.

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